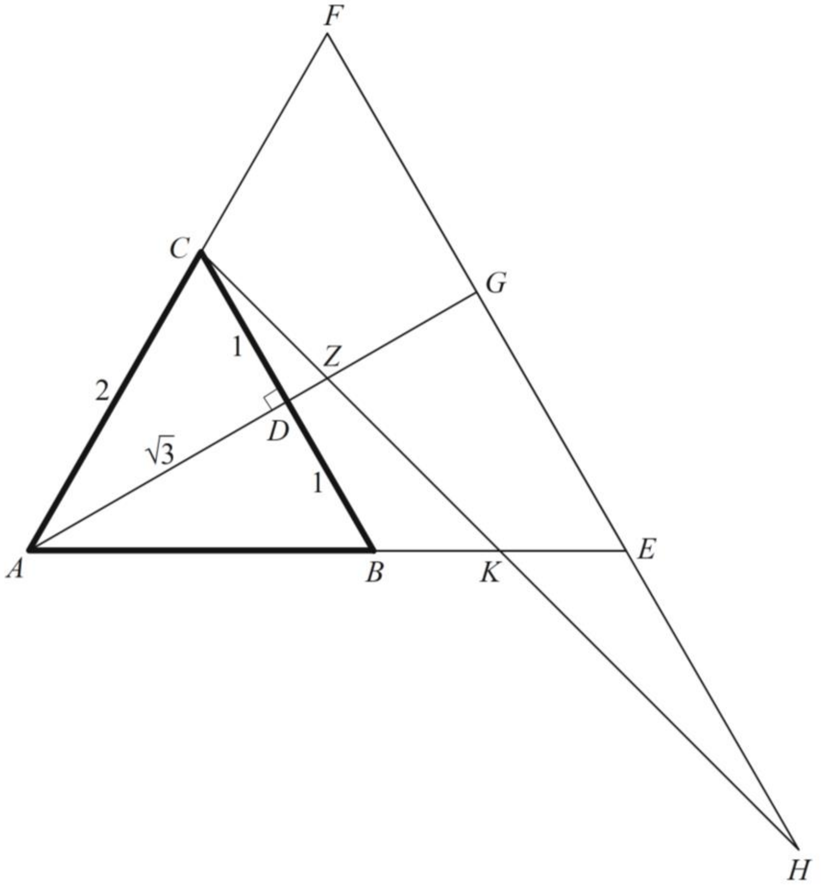
Exam training session on geometry

Question 1.

Let the equilateral triangle ΔABC with sides of length 2 be given.



In triangle ΔABC line segment AD is both an altitude line and a median line. Therefore: $BD = CD = 1$ and $AD = \sqrt{3}$.

The equilateral triangle ΔAEF with sides of length $2\sqrt{3}$ is also given, where E and F lie on the extended lines AB and AC. Line AD intersects EF at G.

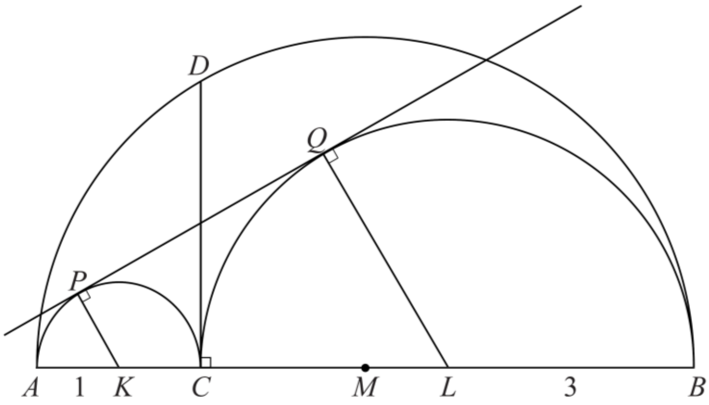
Z is the centroid/median of ΔAEF. The line through points C and Z intersects AE at K and the extended line FE at H.

a. Prove that $DZ = 2 - \sqrt{3}$.

b. Prove that EH has the same length as AB.

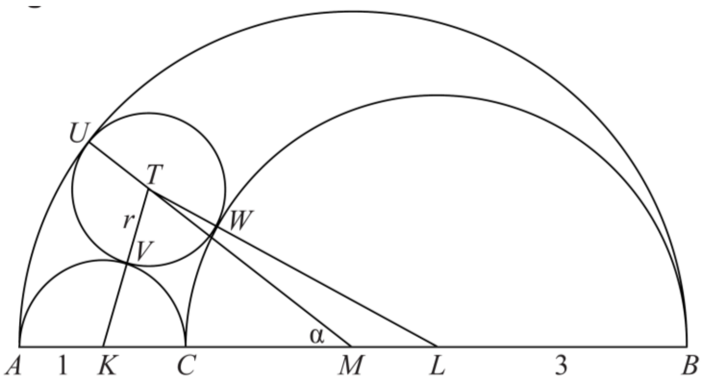
Question 2. Consider a semicircle with diameter AB and radius 4. The centre of the circle is M. Point C is chosen on line segment AB such that $AC = 2$. Line segments AC and AB are diameters of two semicircles in their own right. The radii of two circles are therefore 1 and 3. Points K and L are centres of these two semicircles. All semi-circles lie on the same side of line segment AB.

The line through C perpendicular to AB intersects the largest semicircle at point D. Line segment PQ is the common tangent line of the two smallest semicircles, where P and Q are the tangent points.



10. Prove that CD and PQ have the same length.

There is one circle that is tangent to the three semicircles. This circle has radius r and centre T.



$\angle TMK$ is called α.

11. Use the cosine rule in ΔMKT to prove that

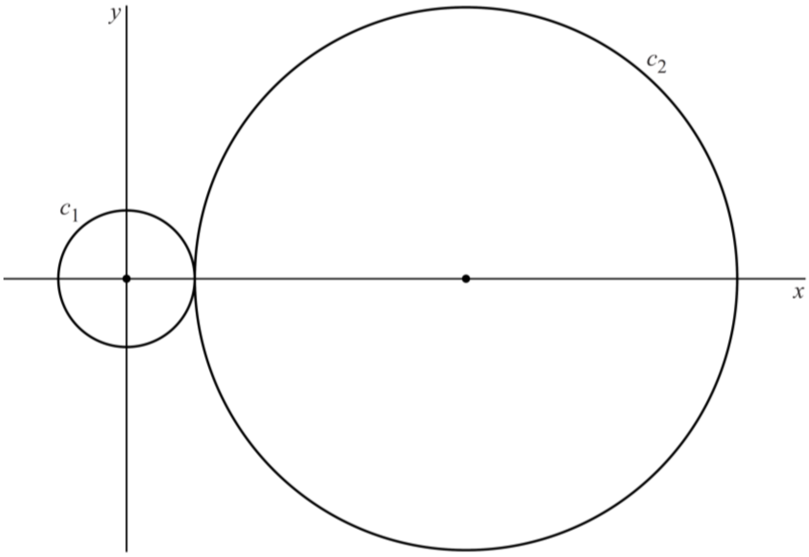
$$\cos(\alpha) = \frac{12 - 5r}{12 - 3r}$$

If the cosine rule is applied to triangle MLT we obtain

$$\cos(\alpha) = \frac{7r - 4}{4 - r}$$

12. Calculate r analytically.

Question 3. Consider circle c_1 with equation $x^2 + y^2 = 9$ and circle c_2 with equation $(x - 15)^2 + y^2 = 144$



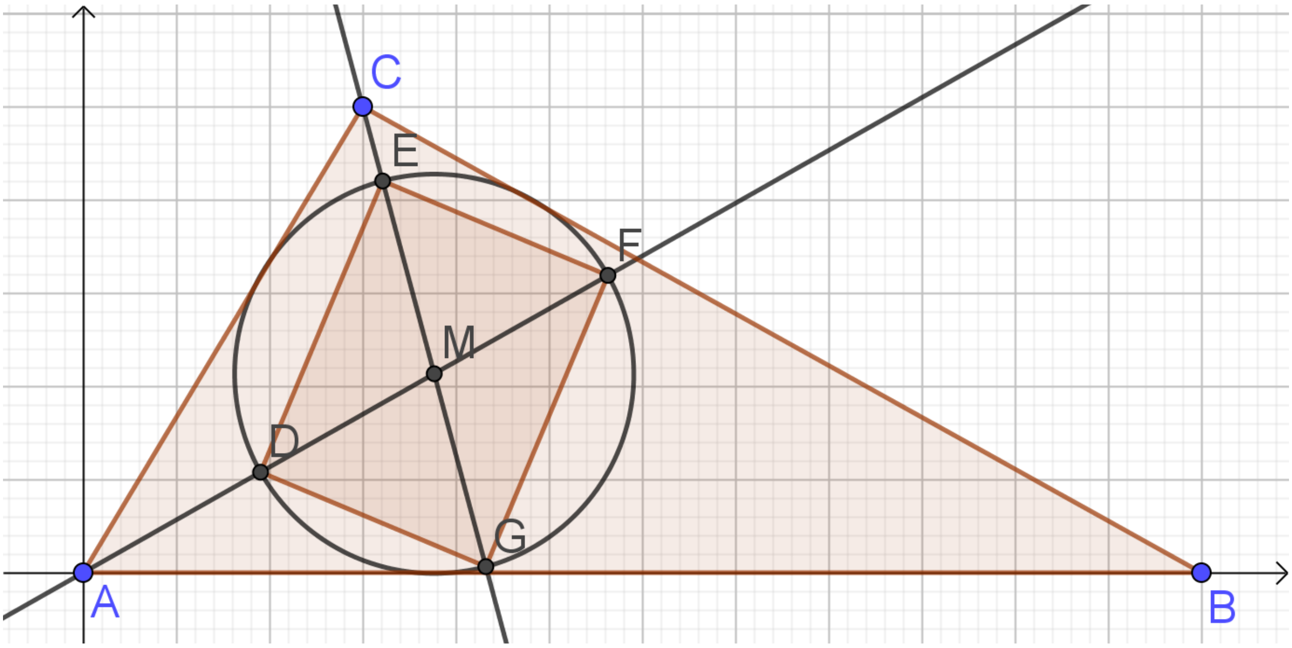
The centre of a circle c_3 lies on the positive y-axis and c_3 is tangent to both circle c_1 and circle c_2.

16. Find an equation for circle c_3

The circle c_1 and c_2 have three tangent lines in common.

17. Determine equations for each of these three tangent lines.

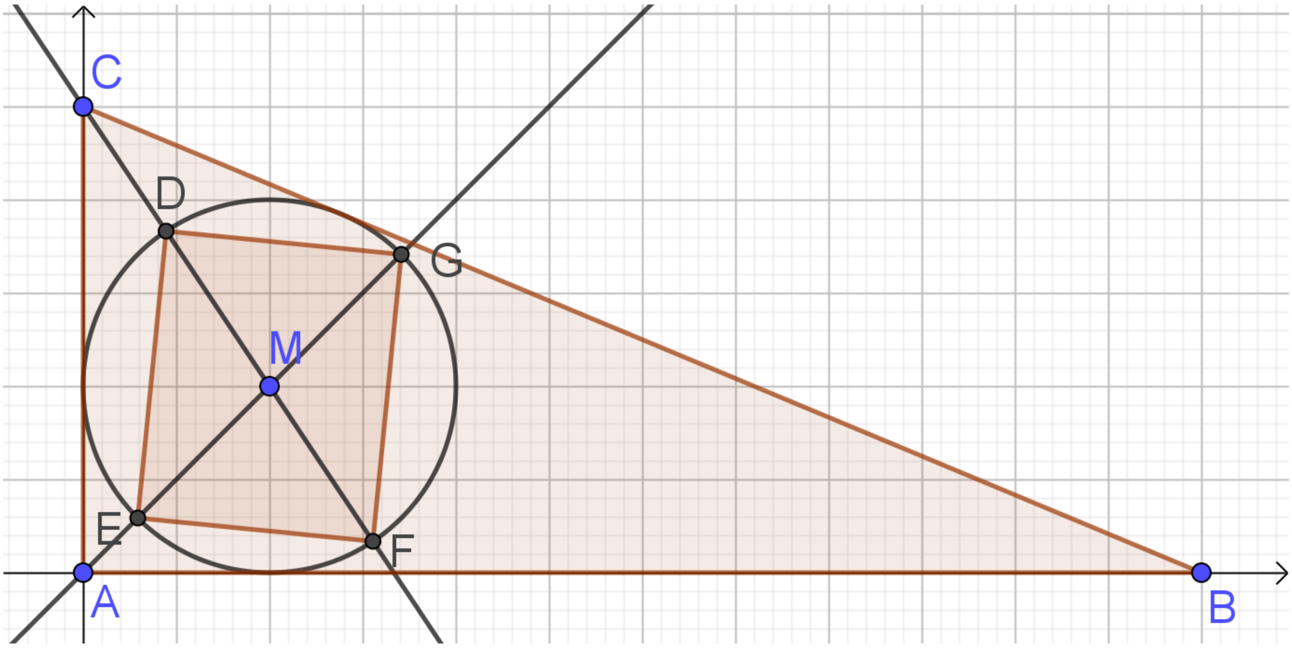
Question 4. Consider an arbitrary triangle ΔABC and its inscribed circle.



The angle bisector of $\angle A$ and the angle bisector of $\angle C$ of the triange intersect the inscribed circle at points D, E, F and G.

a. Prove that $DEFG$ is a rectangle.
b. Prove that $\angle MDE = \frac{1}{4}\alpha + \frac{1}{4}\gamma$
c. Prove that, although $DEFG$ is a rectangle, it cannot be a square.

Question 5. Consider ΔABC with $A(0,0)$, $B(12,0)$ and $C(0,5)$



The angle bisector of $\angle A$ and the angle bisector of $\angle C$ of the triange intersect the inscribed circle at points D, E, F and G.

a. Prove that the inscribed circle is given by $x^2 - 4x + y^2 - 4y = -4$
b. Determine an equation for the angle bisectors of angles $\angle A$ and $\angle C$
c. Prove that the radius of the inscribed circle is equal to 2.
d. Find the coordinates of E